

Continuous Variables_ I

Note Title

9/25/2010

qubit

qutrit

qudit

cv

$d=2$

$d=3$

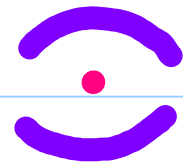
d

∞

qubit:

$\begin{array}{c} \text{---} e \\ \text{---} g \end{array}$

$\begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \\ \text{---} \end{array}$



$\begin{array}{c} \uparrow \\ H \end{array}, \begin{array}{c} \rightarrow \\ V \end{array}$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

σ_1

σ_2

σ_3

σ_0

$$\sigma_{\mu\nu} = \sum_j (-1)^{j\nu} |j+\mu\rangle \langle j|$$

$$\sigma_{00} = I, \quad \sigma_{01} = Z, \quad \sigma_{10} = X, \quad \sigma_{11} = Y$$

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$$

$$Z: |0\rangle, |1\rangle$$

$$X: |+\rangle, |-\rangle$$

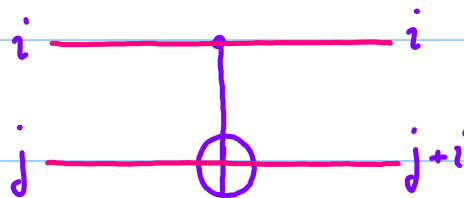
$$|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

$$H = |+\rangle\langle 0| + |-\rangle\langle 1| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad H_{ij} = \frac{1}{\sqrt{2}} (-1)^{ij}$$

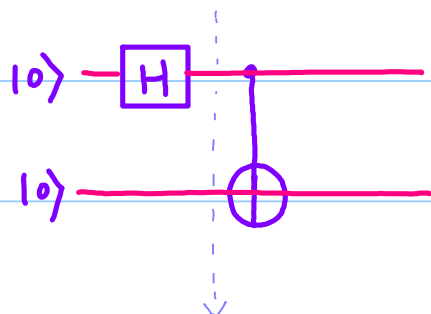
$$H Z H^\dagger = X \rightarrow H^\dagger X H = Z$$

$$CNOT |i, j\rangle = |i, i+j\rangle$$



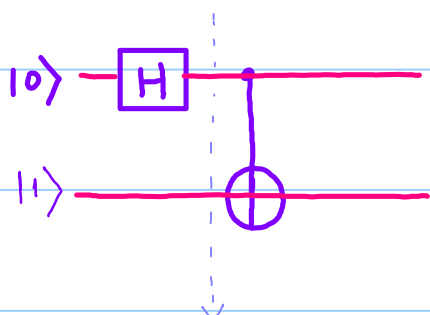
$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle)$$

$$|\phi_\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle)$$



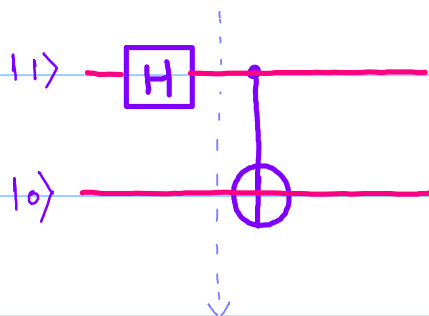
$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = |\psi_+\rangle$$

$$\frac{1}{\sqrt{2}} (|00\rangle + |10\rangle)$$



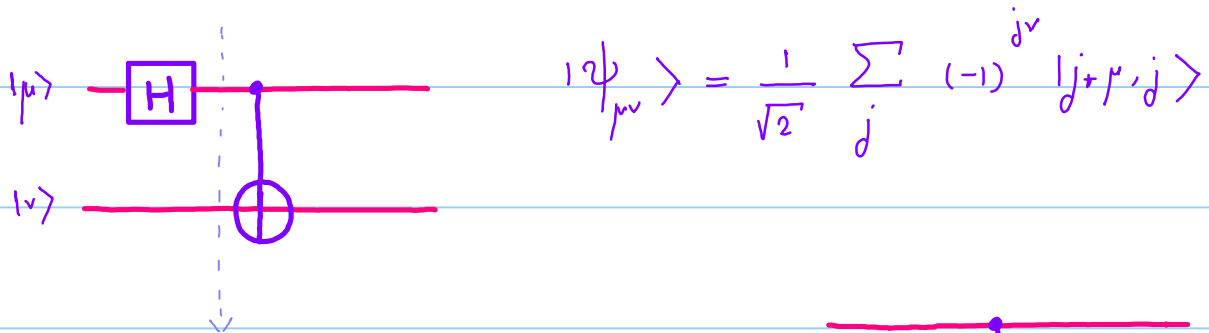
$$\frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)$$

$$\frac{1}{\sqrt{2}} (|01\rangle + |11\rangle)$$

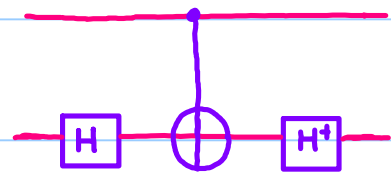


$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

$$\frac{1}{\sqrt{2}} (|00\rangle - |10\rangle)$$



$$CZ = (I \otimes H^\dagger) CX (I \otimes H)$$



Many other things are simple or rather
simple for qubits:

● Separability of bi-partite systems

1) For pure states: $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

$$\text{if } ad - bc = 0 \iff \psi = \phi \otimes \chi$$

2) for mixed states

Given a density Matrix $\rho = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & . & . & . \\ . & . & . & . \\ \rho_{41} & . & . & \rho_{44} \end{bmatrix}$

if $\rho^{T_B} \geq 0 \iff \rho$ is separable.

Peres PPT criteria. peres PRL (1998)

Positive partial Transpose. (PPT).

• Entanglement Measure for bi-partite systems.

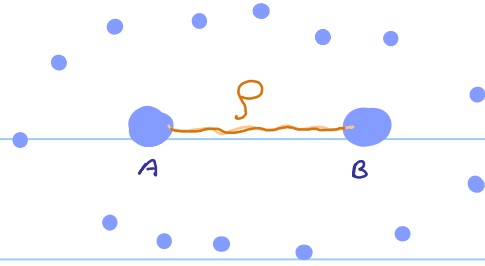
1) for pure states



$$\mathcal{E} = S(\rho) = -\text{tr}_A(\rho \ln \rho)$$

$$\rho = \text{tr}_B(|\psi\rangle\langle\psi|).$$

2) For Mixed states



$$\mathcal{E} = \mathcal{E}(\rho_D)$$

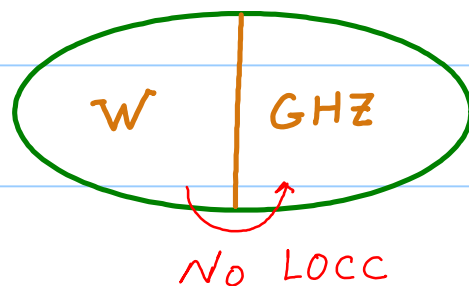
$$\rho_D = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{1 - c^2} & \\ & 1 - \sqrt{1 - c^2} \end{pmatrix}$$

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

$$\lambda_i \text{'s} = \pm \sqrt{\text{eigenvalues of } \rho \tilde{\rho}},$$

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$$

● Three-partite Entanglement



$$|W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle)$$

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

Choi-Matrix: $\mathcal{Z}_\phi := \sum_{ij} \phi(E_{ij}) \otimes E_{ij} \in \mathcal{B} \otimes A$

1) if ϕ trace preserving $\equiv \text{tr}(\phi(\rho)) = \text{tr}(\rho) \rightarrow$

$$\text{tr}(\phi(E_{ij})) = \text{tr}(E_{ij}) = \delta_{ij}$$

$$\rightarrow \text{Tr}_1(\mathcal{Z}_\phi) = I$$

$$\text{if } \text{Tr}(\mathcal{Z}_\phi) = I \longrightarrow \text{Tr} \phi(E_{ij}) = \delta_{ij}$$

$$\rightarrow \text{Tr}(\phi(\rho)) = \text{Tr}(\phi(\sum_{ij} p_{ij} E_{ij})) =$$

$$= \text{Tr}(\sum_{ij} p_{ij} \phi(E_{ij})) = \sum_{ij} p_{ij} \text{Tr}(\phi(E_{ij}))$$

$$= \sum_{ij} p_{ij} \delta_{ij} = \text{Tr}(\rho). \rightarrow \phi \text{ is TP.}$$

$$\text{So } \phi \text{ is TP} \longleftrightarrow \text{Tr}_1 \mathcal{Z}(\phi) = 1$$

$$2) \quad \phi \text{ is unital} \equiv \phi(I) = I$$

$$\rightarrow \text{Tr}_2(\mathcal{Z}_\phi) = \sum_{ij} \phi(E_{ij}) \delta_{ij}$$

$$= \phi(I) = I.$$

Conversely:

$$\text{if } \text{Tr}_2(\mathcal{Z}_\phi) = I \rightarrow \text{Tr}_2\left(\sum_{ij} \phi(E_{ij}) \otimes E_{ij}\right) = \mathbb{I}$$

$$\rightarrow \sum_{ij} \phi(E_{ij}) \otimes \delta_{ij} = \mathbb{I} \rightarrow \phi\left(\sum_i E_{ii}\right) = \mathbb{I} \rightarrow$$

$$\phi(1) = \mathbb{I} \rightarrow \phi \text{ is unital}$$

$$\text{So: unital } \phi \longleftrightarrow \text{Tr}_2(\mathcal{Z}_\phi) = I.$$

● Hermiticity preserving $\equiv$ if $\rho = \rho^\dagger \rightarrow \phi(\rho) = [\phi(\rho)]^\dagger$

you can show that for such a ϕ :

$$[\phi(E_{ij})]^\dagger = \phi(E_{ji})$$

$$\begin{aligned} \text{Then } (\mathcal{Z}_\phi)^\dagger &= \sum_{j'} [\phi(E_{j'})]^\dagger \otimes E_{j'}^\dagger = \\ &= \sum_{j'} \phi(E_{j'}) \otimes E_{j'} = \mathcal{Z}_\phi. \end{aligned}$$

So $HP \ \Phi \longleftrightarrow \text{Hermitian } \mathcal{Z}_\phi$

Finally

Φ is Completely positive $\longleftrightarrow \mathcal{Z}_\phi$ is positive.

● positive Maps on Qubits:

32. [arXiv:quant-ph/0101003](#) [pdf, ps, other]

An Analysis of Completely-Positive Trace-Preserving Maps on 2x2 Matrices

Mary Beth Ruskai, Stanislaw Szarek, Elisabeth Werner

Comments: A significantly expanded version of [quant-ph/0005004](#). 34 pages, 3 figures Final version accepted for publication in Lin. Alg. Appl

Journal-ref: Lin. Alg. Appl. 347, 159-187 (2002)

Subjects: Quantum Physics (quant-ph); Mathematical Physics (math-ph); Operator Algebras (math.OA); Rings and Algebras (math.RA)

Bloch sphere $\rho = w_0 I + \vec{w} \cdot \vec{\sigma}$

Φ : A completely positive map.

$$\Phi(\rho) = w_0 I + (w_0 \vec{t} + T \vec{w}) \cdot \vec{\sigma}$$

$$\begin{pmatrix} w_0 \\ \vec{w} \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ \vec{t} & T \end{pmatrix}}_T \begin{pmatrix} w_0 \\ \vec{w} \end{pmatrix}$$

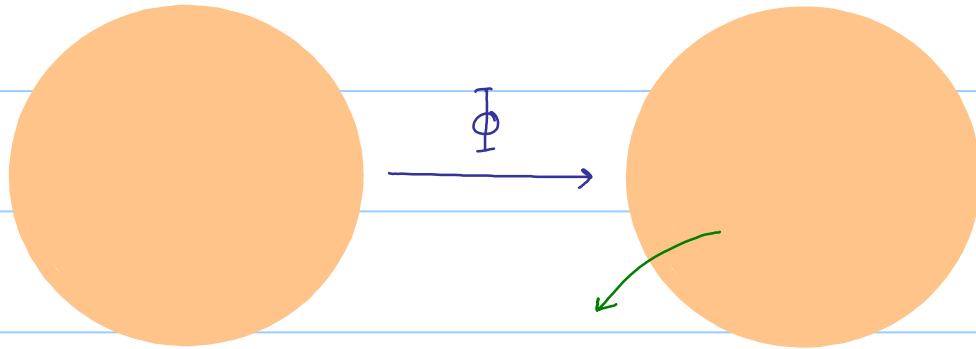
King & Ruskai : Change of Basis

[13] C. King and M.B. Ruskai, "Minimal entropy of states emerging from noisy quantum channels" *IEEE Trans. Info. Theory* **47**, 192-209 (2001).

$$T \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{bmatrix} \quad (\lambda_i \pm \lambda_j) \leq (1 \pm \lambda_k)$$

↑
Complete positivity.

Bloch sphere



$$\left(\frac{x_1 - t_1}{\lambda_1} \right)^2 + \left(\frac{x_2 - t_2}{\lambda_2} \right)^2 + \left(\frac{x_3 - t_3}{\lambda_3} \right)^2 = 1.$$

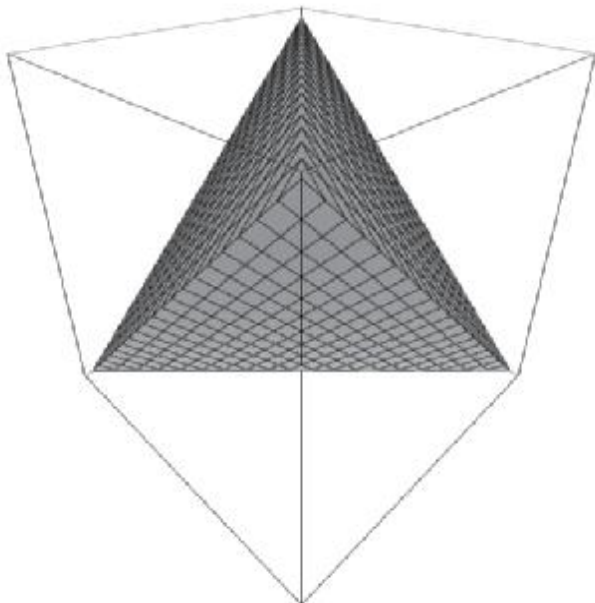
$$\Phi(\rho) = u \Phi_{t,1}(v \rho v^\dagger) v^\dagger$$

$$(\lambda_1 \pm \lambda_2) \leq 1 \pm \lambda_3 \rightarrow \lambda_1 + \lambda_2 + \lambda_3 \leq 1$$

$$\lambda_1 - \lambda_2 + \lambda_3 \leq 1$$

$$\lambda_1 + \lambda_2 - \lambda_3 \leq 1$$

$$\lambda_1 - \lambda_2 - \lambda_3 \leq 1$$



Qudits

— $d-1$

— 2

— 1

— 0

qudit: $|0\rangle, |1\rangle, \dots, |d-1\rangle$

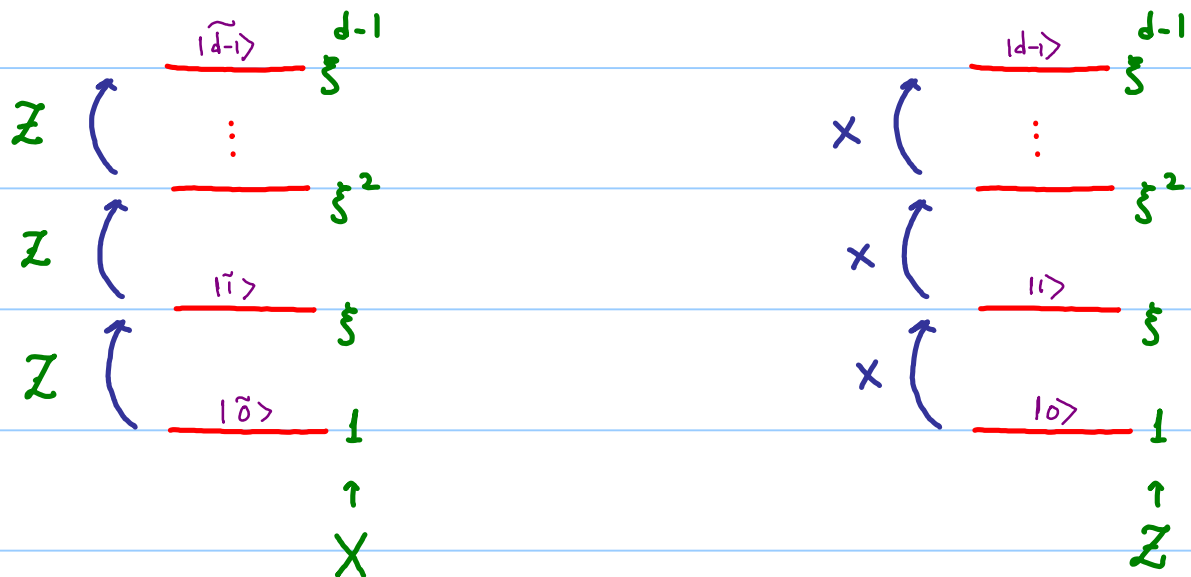
$$(-1)^2 = 1 \rightarrow \xi^d = 1 \quad \xi = e^{2\pi i/d}$$

$$X = \sum_{i=0}^{d-1} |i+1\rangle \langle i| \quad Z = \sum_{i=0}^{d-1} \xi^i |i\rangle \langle i|$$

$$XZ = \xi ZX \quad X^d = I \quad Z^d = I.$$

eigenvalues of $X, Z = \{1, \xi, \xi^2, \dots, \xi^{d-1}\}$

$$Z|i\rangle = \xi^i |i\rangle \quad X|\tilde{i}\rangle = \xi^i |\tilde{i}\rangle$$



$$|\tilde{i}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \xi^{ij} |j\rangle$$

$$H_{ij} = \frac{1}{\sqrt{d}} \xi^{ij}$$

Many things for qudits are NOT simple:

- No Bloch sphere.
- Separability

- pure state:

$$|\psi\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{02}|02\rangle + \\ \psi_{10}|10\rangle + \psi_{11}|11\rangle + \psi_{12}|12\rangle + \\ \psi_{20}|20\rangle + \psi_{21}|21\rangle + \psi_{22}|22\rangle$$

$$|\phi_1\rangle \otimes |\phi_2\rangle = (a_1|0\rangle + b_1|1\rangle + c_1|2\rangle) \otimes (a_2|0\rangle + b_2|1\rangle + c_2|2\rangle)$$

6 parameters instead of 9. $\rightarrow$ 3 equations.

$$\psi_{00} \psi_{11} \psi_{22} = \psi_{01} \psi_{12} \psi_{20} = \psi_{02} \psi_{21} \psi_{10} \\ = \psi_{01} \psi_{10} \psi_{22} = \psi_{00} \psi_{12} \psi_{21} = \psi_{02} \psi_{11} \psi_{20}.$$

Another criteria: $\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|)$.

if $|\psi\rangle$ is separable $\rightarrow \rho_A = |\phi_1\rangle\langle\phi_1| \rightarrow \rho_A^2 = \rho_A$.

- Mixed Case:

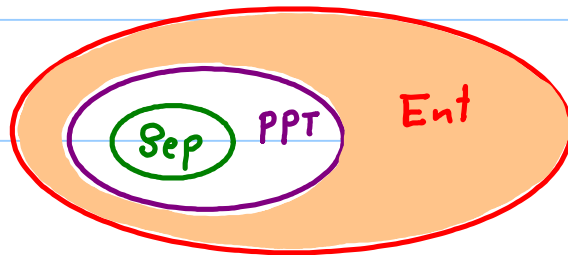
Seperable $\longrightarrow$ PPT.

↑
True in every dimension.

NPT $\longrightarrow$ Entangled.

But what if a state is PPT?

$\longrightarrow$ Bound Entangled.



- Entanglement

- pure

$$|\psi\rangle_{AB} \longrightarrow E = -\text{tr}(\rho_A \ln \rho_A)$$

- Mixed

$$\rho_{AB} \longrightarrow ?$$

$\rho_{AB}^{T_B}$

= partial Transpose $\rightarrow$

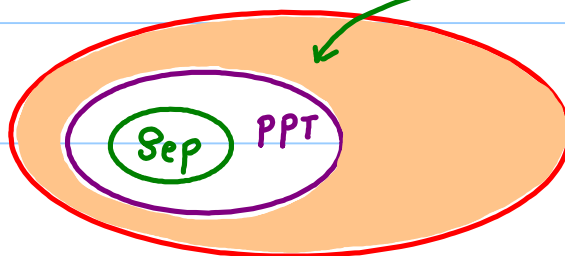
$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \\ & & & & \lambda_d \end{bmatrix}$$

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_B}\|_1 - 1}{2}$$

$$\text{where } \|X\|_1 = \sum_{i=1}^d |\lambda_i|$$

$$\begin{aligned} \mathcal{N}(\rho) &= \frac{1}{2} \left(\sum_{i=1}^d (\lambda_i^+ - \lambda_i^-) - \sum_{i=1}^d (\lambda_i^+ + \lambda_i^-) \right) \\ &= - \sum_{i=1}^d \lambda_i^- > 0 \end{aligned}$$

$\mathcal{N}(\rho) > 0 \longrightarrow$ Negative eigenvalue



● Positive maps on qubits ?

Analysis of complete positivity conditions for quantum qutrit channels

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(Dated: December 15, 2008)

Generalized Bloch sphere:

$$\rightarrow \rho = \frac{1}{d} (I + \vec{w} \cdot \vec{T})$$

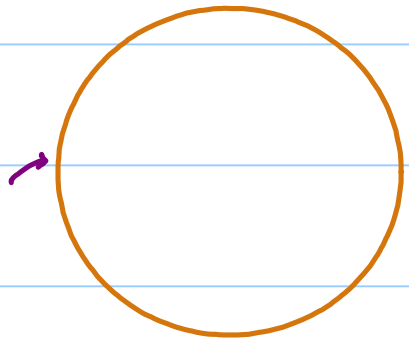
T basis of Hermitian Traceless Matrices for $SU(d)$.

Normalization: $\text{Tr}(T_i T_j) = d(d-1) \delta_{ij}$

$$\text{Then } \text{Tr}(\rho^2) \leq 1 \rightarrow \frac{1}{d^2} (d + d(d-1) w \cdot w) < 1$$

$$\rightarrow \vec{w} \cdot \vec{w} \leq 1.$$

7-dim.
sphere



But Not every $w \rightarrow$ a positive ρ . !!

for qutrits: $\rho = \frac{1}{3} (1 + \sqrt{3} n \cdot \lambda)$

$\lambda_1, \dots, \lambda_8$ (Gellman Matrices)

$$\text{Tr}(\lambda_i \lambda_j) = 2 \delta_{ij}$$

$$\lambda_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \lambda_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \quad \dots$$

$$\lambda_i \lambda_j = d_{ijk} \lambda_k.$$

$$\rho = \frac{1}{3} (1 + \sqrt{3} \vec{n} \cdot \vec{\lambda})$$

pure states: $\vec{n} \cdot \vec{n} = 1$ $\vec{n} * \vec{n} = \vec{n}$

where $(A * B)_i = d_{ijk} A_j B_k$.

$$\Phi : \vec{n} \mapsto \vec{n}' = \Lambda \vec{n} + \vec{t}, \quad (15)$$

where $\Lambda = \text{diag}\{\Lambda_1, \dots, \Lambda_8\}$ consists of 8 damping coefficients and $\vec{t}$ is an eight dimensional translation. The image of the set of pure states under this transformation is

$$\sum_{i=1}^8 \left(\frac{n'_i - t_i}{\Lambda_i} \right)^2 = 1, \quad (16)$$

together with the condition for *-product $\vec{n} * \vec{n} = \vec{n}$

$$\frac{n'_i - t_i}{\Lambda_i} = d_{ijk} \frac{n'_j - t_j}{\Lambda_j} \frac{n'_k - t_k}{\Lambda_k}. \quad (17)$$

$$1 - \Lambda_8 + \frac{3}{2}(\Lambda_4 - \Lambda_5) \geq 0,$$

$$1 - \Lambda_8 - \frac{3}{2}(\Lambda_4 - \Lambda_5) \geq 0,$$

$$1 - \Lambda_8 + \frac{3}{2}(\Lambda_6 - \Lambda_7) \geq 0,$$

$$1 - \Lambda_8 - \frac{3}{2}(\Lambda_6 - \Lambda_7) \geq 0, \quad (21)$$

$$1 - \Lambda_8 + \frac{3}{2}(\Lambda_1 - \Lambda_2) + \frac{3}{2}(\Lambda_8 - \Lambda_3) \geq 0,$$

$$1 - \Lambda_8 - \frac{3}{2}(\Lambda_1 - \Lambda_2) + \frac{3}{2}(\Lambda_8 - \Lambda_3) \geq 0.$$